# Dynamic Term Structure Modeling

The Fixed Income Valuation Course


Sanjay K. Nawalkha

Natalia A. Beliaeva

Gloria M. Soto


FixedIncomeRisk.com

- **Dynamic Term Structure Modeling. The Fixed Income Valuation Course.** Sanjay K. Nawalkha, Natalia A. Beliaeva, Gloria M. Soto, 2007, Wiley Finance.
  - **Chapter 3: Valuing Interest Rate and Credit Derivatives: Basic Pricing Frameworks**

- Goals:
  - Introduce basic pricing frameworks for valuing interest rate and credit derivatives.
  - Describe the features of these derivatives and identify the underlying relationships among derivative prices.
  - Introduce a new taxonomy for term structure models that classifies all models either fundamental models or preference-free models.

# Valuing Interest Rate and Credit Derivatives: Basic Pricing Frameworks

- **Pricing Frameworks for Valuing Time Deposit and Treasury Futures**

- **Pricing Frameworks for Valuing Basic Interest Rate Derivatives and Credit Derivatives**

- **A New Taxonomy of Term Structure Models**

# A New Taxonomy of Term Structure Models

- Introduction
- Fundamental Models
- Preference-Free Models
- Comparative Analysis
- Conclusion

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# Introduction

- The origins of term structure models can be traced back to a footnote in the Nobel prize-winning work of Merton [1973], in which he related the dynamics of the bond price to that of the instantaneous default-free short rate.
  - Like other famous footnotes in finance, this footnote was extended in many directions, eventually leading to the whole sub-field of finance known as “term structure models.”
  - These models translate the uncertainty in interest rates into the uncertainty in traded securities in an arbitrage-free setting, thus allowing a rational determination of the prices of financial derivatives whose value depends upon the evolution in interest rates.

# Introduction

- Though Merton conceived the idea of term structure modeling, Vasicek can be called the real father of term structure theory.

- From the earliest terms structure models to the latest innovations, all use the basic arbitrage-free framework introduced by Vasicek [1977].

# Introduction

- Though Vasicek is now associated with the specific example of the Ornstein-Uhlenbeck process for the instantaneous short rate, his original paper can be used to model virtually any Markovian term structure model in which zero-coupon yields are the underlying drivers of uncertainty.
  - For example, all short rate models, from the square root model of Cox, Ingersoll, and Ross (CIR) [1985] to the multifactor ATSMs of Dai and Singleton [2000] are solved using the partial differential equation known as the term structure equation originally derived by Vasicek.

# Introduction

- Additional restrictions can be imposed on the market price of interest rate risk (reward for bearing risk) using the equilibrium frameworks developed CIR and others.
  - Of course, these restrictions are consistent with Vasicek's term structure equation since "absence of arbitrage" conditions are weaker than the "equilibrium" conditions.

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# Fundamental Models

- The Vasicek and CIR models are fundamental term structure models (TSMs), which like all other fundamental TSMs, share two properties, as follows:
  - A time-homogenous short rate process.
  - An explicit specification of the market prices of risks.
- Fundamental TSMs value default-free zero-coupon bonds using the information related to investors' risk aversion and expected movements in the interest rates, similar to how fundamental equity models value stocks using the information related to earnings, systematic risk, and growth rate in earnings.

# Fundamental Models

- A variety of multifactor fundamental TSMs have been derived in the past decade, chief among them being models in the affine and quadratic classes.
  - Fundamental models are applied by traders interested in relative arbitrage among default-free bonds of different maturities.
- These models are estimated using econometric techniques such as maximum likelihood, generalized method of moments, simulated method of moments, using time-series data on zero-coupon yields.
- The intrinsic model prices implied by fundamental models may or may not converge to the market prices of bonds.

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# Preference-Free Models

- In contrast to fundamental models, preference-free models do not require explicit specifications of the market prices of risks for valuing bonds and interest rate derivatives.
  - Hence, valuation can be done without knowing the risk-preferences of the market participants under preference-free models.
  - We will consider three types of preference-free TSMs given as single-plus, double-plus, and triple-plus models.

# Preference-Free Models

- We show that a preference-free single-plus TSM exists corresponding to every fundamental TSM.
- The only difference between the fundamental TSM and the corresponding preference-free single-plus TSM is that the former requires an explicit specification of the market prices of risks (MPRs), while the latter does not require MPRs for valuing bonds and interest rate derivatives.
  – In effect, since the latter does not require MPRs, it is consistent with general, non-linear specifications of MPRs, which allows it to fit better with the market prices of bonds and interest rate derivatives.

# Preference-Free Models

- Though preference-free single-plus models could imply arbitrage possibilities using only one or two factors, these models are virtually free of arbitrage with higher number of factors, as the model bond prices become indistinguishable from the observed bond prices with more factors.

# Preference-Free Models

- The risk-neutral stochastic processes of the state variables under any single-plus TSM are identical in form to the risk-neutral stochastic processes of the state variables under the corresponding fundamental TSM.
  - However, the empirical estimates of the risk-neutral parameters are generally different under these two models, as the latter model imposes restrictive functional forms on the specifications of MPRs.
  - The restrictive MPRs under the latter model also imply that the stochastic processes of the state variables under these two models are different under the physical measure.

# Single-Plus Term Structure Models

- The trick to the derivation of a single-plus TSM corresponding to a given fundamental TSM is to specify the stochastic bond price process exogenously using the same form of volatility function used under the given fundamental model.
- The exogenous stochastic bond price process is then combined with an exogenously given solution of the time-zero bond prices or forward rates, which leads to a time-homogenous risk-neutral short rate process.
  - By fitting the prices implied by the single-plus TSM to the time-zero observed prices of bonds and interest rate derivatives, the risk-neutral parameters and state variable values are determined.

# Single-Plus Term Structure Models

– Since single-plus TSMs obtain the short rate process endogenously using an exogenous stochastic bond price process, these models allow independence from the MPRs.

– On the other hand, since fundamental TSMs assume the short rate process under the physical measure, and since the short rate does not trade, these models require explicit dependence on the MPRs for obtaining valuation formulas of bonds and interest rate derivatives.

# Double-Plus Term Structure Models

- The preference-free double-plus TSMs are different from the corresponding fundamental TSMs in two ways.
- These models are not only free of the MPR specifications - similar to the single-plus models - but they also allow the model bond prices to exactly fit the initially observed bond prices.
  - Unlike the single-plus TSMs, that may require multiple factors to match the model prices with the observed prices, the double-plus TSMs can allow an exact fit even using a single factor.
  - The initially observed bond prices are used as an input under the double-plus TSMs.

# Double-Plus Term Structure Models

- These models exactly fit the initially observed bond prices by allowing time-inhomogeneity in the drift of the risk-neutral short rate process.
- This is unlike the single-plus models, which require a time-homogenous drift for the risk-neutral short rate process.

- Examples of double-plus TSMs include the models by Ho and Lee [1986], Hull and White [1990], Heath, Jarrow, and Morton (HJM) [1992], and Brigo and Mercurio [2001].

# Double-Plus Term Structure Models

- Though double-plus models can be derived corresponding to all fundamental TSMs (or corresponding to all single-plus TSMs), the vice-versa is not necessarily true.
  - For example, no fundamental TSM or single-plus TSM may exist corresponding to the non-Markovian double-plus HJM models.

# Triple-Plus Term Structure Models

- The preference-free triple-plus TSMs are different from the corresponding fundamental TSMs in three ways.
- Unlike the fundamental models, but similar to single-plus and double plus models, these models are free of the MPR specifications.
- Unlike the fundamental and single-plus models, but similar to double plus models, these models allow an exact fit with the initially observed bond prices.

# Triple-Plus Term Structure Models

- However, unlike the fundamental, single-plus, and double-plus models, which all require a time-homogenous specification of volatilities, the triple-plus TSMs allow time-inhomogenous volatilities (i.e., time-inhomogenous short rate volatility and/or time-inhomogenous forward rate volatilities).
- Examples of triple-plus TSMs include extensions of the models of Hull and White [1990], Black, Derman, and Toy [1990], and Black and Karasinski [1991] with time-inhomogenous volatilities, and versions of LIBOR market model with time-inhomogenous volatilities (see Brigo and Mercurio [2001, 2006] and Rebonato [2002]).

# Triple-Plus Term Structure Models

- These models originated from the work of practitioners interested in pricing exotic interest rate derivatives, relative to the pricing of some plain-vanilla derivative benchmarks, such as caps and/or swaptions.
- The triple-plus models are motivated by the need to exactly fit the initial prices of the chosen set of plain-vanilla derivatives, in addition to exactly fitting the initial bond prices.
  - However, the triple-plus models require a high numbers of parameters to obtain an exact fit with the chosen plain-vanilla derivative instruments, and may suffer from the criticism of “smoothing.”

# Triple-Plus Term Structure Models

  - Our Bayesian priors regarding the usefulness of various classes of term structure models are best depicted using an inverted U- curve that plots the usefulness of the TSMs against the number of plusses, with zero-plus denoting the fundamental TSMs.

- Going from zero-plus to one-plus, the marginal benefit may be significant, as allowing flexibility in the specifications of MPRs is known to significantly enhance the performance of term structure models (see Duffee [2002] and Duarte [2004]).

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# Comparative Analysis

- Hence, allowing TSMs to be completely independent of MPRs, makes these models consistent with very general, non-linear MPRs, and allows more realistic stochastic processes under the physical measure.
  - For example, as shown in chapter 8, the two-factor single-plus affine model, or the A2(2)+ model, can allow negative unconditional correlation between the two state variables under the physical measure, even though it must disallow negative correlation under the risk-neutral measure.
  - In contrast, the two-factor fundamental affine model, or the A2(2) model must disallow negative correlation under both the physical measure and the risk-neutral measure (see Dai and Singleton [2000]).

# Comparative Analysis

- The bond pricing formulas and the entire analytical apparatus for pricing derivatives is identical under the fundamental TSMs and single-plus TSMs, except that the empirical estimates of the risk-neutral parameters may be different under these two classes of models.
- Though single-plus models allow a time-homogenous short rate process, these models may not allow a good fit between the model bond prices and the observed bond prices, when using a very small number of factors (e.g., only one or two factors).

# Comparative Analysis

- Hence, double-plus models may be useful as these models allow exact calibration to the initially observed bond prices, even with a low number of factors, by allowing a time-inhomogenous short rate process.
  - Further, since double-plus models are preference-free, they share the same advantage of the single-plus models over the fundamental models, as mentioned above.

# Comparative Analysis

- However, as the number of factors increase, the advantage of having a time-homogeneous short rate process may dominate the disadvantage of not exactly fitting the initially observed bond prices, using single-plus models. This is because with a higher number of factors, single plus models can fit the observed bond prices, almost perfectly, if not exactly.
- Though the triple-plus models may at first appear more general than single-plus or double-plus models, these models suffer from the criticism of “smoothing.”

# Smoothing

- In the discussion to follow, we define the term "smoothing" to imply fitting financial models to a set of observed prices without an underlying economic rationale.
  - The concept of smoothing is different from overfitting in that the former implies fitting without an economic rationale, while the latter implies fitting based upon some economic rationale, but using more parameters than needed to obtain a good fit.
  - Smoothing may overlook some important relationships that could potentially be modeled endogenously, while overfitting fits to the noise present in the data.

# Smoothing

- In other words, smoothing allows the modeler to ignore some important economic relationships by making entirely ad-hoc adjustments to fit the model to observed prices (thus, fail to deal with the misspecification error caused by some hidden variables), while overfitting allows the modeler to invent economic relationships that don’t exist but are artifacts of the noise present in the observed prices.

# Smoothing

- A simple example of smoothing is using the Black and Scholes model for pricing equity call options of different strikes, and using different volatilities corresponding to different strikes to fit the "smile" with a third-order polynomial function.
  - If the dynamics of the smile are not modeled based on some economic fundamentals, then a trader may not know why and how the option smile changes over time.
  - The option smile obviously represents some systematic economic factor(s), but the incorporating these factor(s) into the option prices is beyond the scope of the Black and Scholes model.

# Smoothing

- Perhaps, a stochastic volatility/jump model is needed to fit the smile. Yet, if traders continue to use the Black and Scholes model to price options by adjusting the implied volatilities across different strikes to fit the smile using a third-order polynomial, then they are “smoothing.”
- Smoothing basically allows the option trader to price an option of a given strike, given the observed prices of options with strikes surrounding the given strike.
- However, traders can achieve such smoothed prices even by performing a giant Taylor series expansion, without any knowledge of stochastic processes that drive the stock price movements.

# Smoothing

- Similarly, it would be wise to be aware of the dangers of smoothing while considering triple-plus TSMs with a high degree of time-inhomogeneity in the volatility process.
- Though some level of smoothing is present even under the double-plus models, the extent of smoothing under triple-plus models can make these models highly unreliable.
- The origins of time-inhomogenous volatilities as smoothing variables can be traced to the extended versions of the models of Black, Derman, and Toy [1990], Black and Karasinski [1991], and Hull and White [1990].

# Smoothing

- Though practitioners have mostly discarded these earlier generation models, triple-plus versions of the LIBOR market models remain quite popular.
- Rebonato [2002] recognizing the danger of this approach, recommends a three-step process that puts most of the burden of capturing the forward rate volatilities on the time-homogenous component of the forward rate volatilities (see chapter 11).

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# Conclusion

- Though various chapters discuss the fundamental, single plus, and double-plus TSMs in the affine and quadratic class, we limit our attention to the triple plus TSMs only to the case of Vasicek+++ model and the LIBOR market model.
- These models may not be as useful as deemed by their users, given the high degree of “smoothing” resulting from two sources of time-homogeneity, one required to fit the initial bond prices, and the other required to fit the given set of plain vanilla derivative prices.
- On the other hand, fundamental TSMs may be too narrowly defined, due to the restrictive assumptions about the market prices of risks.

# Conclusion

- The single-plus may offer the best of both worlds, allowing preference-free pricing that appeals to practitioners interested in calibration, as well as a time-homogeneous short rate process that appeals to the academics.

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