Interest Rate Risk Modeling
The Fixed Income Valuation Course

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- **Interest Rate Risk Modeling : The Fixed Income Valuation Course.** Sanjay K. Nawalkha, Gloria M. Soto, Natalia K. Beliaeva, 2005, Wiley Finance.
  - **Chapter 11:**
    Duration Models for Default-Prone Securities

- **Goals:**
  - Learn to price and to calculate the duration of default-prone zero-coupon bonds under Merton framework.
  - Learn to price and to calculate the duration of default-prone coupon bonds using the first passage models.

Chapter 11:
Duration Models for Default-Prone Securities

- Introduction
- Pricing and Duration of a Default-Free Zero-Coupon Bond Under The Vasicek Model
- The Asset Duration
- Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework
- Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

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Introduction

- This chapter we are going to turn our attention to **default-prone securities** such as corporate bonds and stocks.

- It's generally assumed in the fixed-income literature that corporate bonds have a lower duration than the equivalent default-free bonds.

- The presence of default risk shortens the expected bond maturity, hence, reduces the duration of the corporate bond. Though intuitively appealing, this reasoning is **flawed**.

Introduction

- Though the possibility of default reduces the expected maturity, we note that the duration of the corporate bond is about five times the duration of the default-free bond.

- **Corporate bond duration is inextricably linked with the duration of the underlying assets of the firm.**

- If the duration of assets is very high, then the duration of the corporate bond can be higher than the duration of the equivalent default-free bond.

Introduction

- The duration of a firm's assets can be given as a **weighted average** of the durations of its non-dividend paying stock and the zero-coupon bond, as follows:

$$D_{\upsilon} = \frac{S(t)}{V(t)} D_s + \frac{D(t,T)}{V(t)} D_d \qquad (11.1)$$

where S(t) is the market value of the stock, D(t,T) is the market value of corporate zero-coupon bond or debt maturing at time T, V(t)=S(t)+D(t,T), is the value of the firm's assets, and D are the durations.

Introduction

- One of the main results found by Chance(1990) is that the duration of default-prone zero-coupon bond is always positive and its magnitude is less than its maturity.
- Implicitly, the duration of the corresponding **non-dividend paying stock must always be negative**, since the duration of the underlying assets of the firm is assumed to be zero. Therefore the duration of the stock is as follow:

$$D_s = -\frac{D(t,T)}{S(t)} D_d \qquad (11.2)$$

Introduction

- The negative stock duration in equation 11.2 implies a wealth transfer from the bondholders of a firm to its stockholders due to an increase in the nominal interest rate.

- However, it’s **inconsistent** with the empirical findings. Stock values fall when interest rates rise and duration values of stocks are generally positive.

Introduction

- Positive durations for stocks imply that the asset durations must be positive in equation 11.1.
- This also implies that the duration of the default-prone zero-coupon bond is higher than when the asset duration is assumed to be zero.
- In this chapter, we allow positive durations for stocks.

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Pricing and Duration of a Default-Free Zero-Coupon Bond Under the Vasicek Model

- Vasicek (1977) assumes a mean reverting Ornstein-Uhlenbeck process for the instantaneous short rate of the form:

$$dr(t) = \alpha\left(m - r(t)\right)dt + \sigma dZ(t) \qquad (11.3)$$

where r(t) is the instantaneous short rate at time t, m is the long-term mean to which r reverts at a speed α, σ is the volatility coefficient and dZ(t) is the standard Wiener process for the short rate.

Pricing and Duration of a Default-Free Zero-Coupon Bond Under the Vasicek Model

- Vasicek obtained the following equation for the bond price at time t maturing T periods hence:

$$P(t,T)=e^{A(t,T)-B(t,T)r(t)} \qquad (11.4)$$

where

$$B(t,T)=\frac{1-e^{-\alpha(T-t)}}{\alpha}$$

$$A(t,T)=\left(m+\frac{\sigma\gamma}{\alpha}-\frac{\sigma^2}{2\alpha^2}\right)[B(t,T)-(T-t)]-B(t,T)^2\frac{\sigma^2}{4\alpha}$$

Pricing and Duration of a Default-Free Zero-Coupon Bond Under the Vasicek Model

- The stochastic bond price process consistent with the above equation is given as:

$$\frac{dP(t,T)}{P(t,T)} = \left(r(t) + \gamma\ B(t,T)\right)dt - B(t,T)dZ(t) \qquad (11.5)$$

- The relative basis risk of the default-free-zero-coupon bond using equation 11.4 can be given as $-[\partial P(t,T) / \partial r(t)] / P(t,T)$, which defines the duration of the bond under the Vasicel model, given as:

$$D_p = -\left(\frac{\partial P\left(t,T\right)}{\partial r\left(t\right)}\right) / P(t,T) = B(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha} \qquad (11.6)$$

Pricing and Duration of a Default-Free Zero-Coupon Bond Under the Vasicek Model

- To make equation 11.4 exactly equals the observable price P(0,T) at time t=0, one may calibrate the Vasicek model by using the deterministically changing mean, m(t), instead of m in equation 11.3.

- Allowing the long-term mean m of the short rate in 11.3 to become a time-dependent function, which is given as follows:

$$m(t) = \frac{1}{\alpha}\left(\frac{\partial f(0,t)}{\partial t} + \alpha f(0,t) + \frac{\sigma^2}{2\alpha}\left[1 - e^{-2\alpha t)}\right] - \gamma\ \sigma\right) \qquad (11.7)$$

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The Asset Duration

- The asset price **cannot** be given as a deterministic function of the short rate, and hence one cannot obtain the duration of the assets by taking the partial derivative of the asset price with respect to the short rate.

- However, the **conditional expected change in the asset return** can be measured for a given change in the short rate using a statistical regression technique.

The Asset Duration

- Using this insight, we define the asset duration as the sensitivity of the asset return to changes in the short rate.

- Assume that a firm's asset price dynamics are given by the following diffusion process:

$$\frac{dV(t)}{V(t)} = \mu \, dt + \sigma_v \, dZ_v(t) \qquad (11.8)$$

The Asset Duration

- Now consider a time-series regression of the asset returns on changes in the short rate given as follows:

$$\frac{dV(t)}{V(t)} = a - D_v \; dr(t) + e_t \qquad (11.9)$$

where a is the intercept and - D_v is the slope coefficient of the regression, where D_v is the duration of the firm's assets.

The Asset Duration

- Using the definition of a regression slope coefficient, the asset duration D_v is given as follows:

$$D_v = -\frac{Cov\left(\frac{dV(t)}{V(t)}\, dr(t)\right)}{Var\left(dr(t)\right)} \qquad (11.10)$$

- Using equations 11.3 and 11.8, the asset duration can be parameterized as follows:

$$D_v = -\frac{(\sigma\ \sigma_v\ \rho)dt}{\sigma^2 dt} = -\frac{\sigma_v\ \rho}{\sigma} \qquad (11.11)$$

where, ρ, the correlation between the two Wiener processes *dZ(t)* and $dZ_v(t)$

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The Asset Duration

- Since,
  - The short rate process and the asset price process are assumed to have constant volatility parameters
  - The correlation between these two processes is also constant

- The duration of the firm's assets is obtained as a **constant**.

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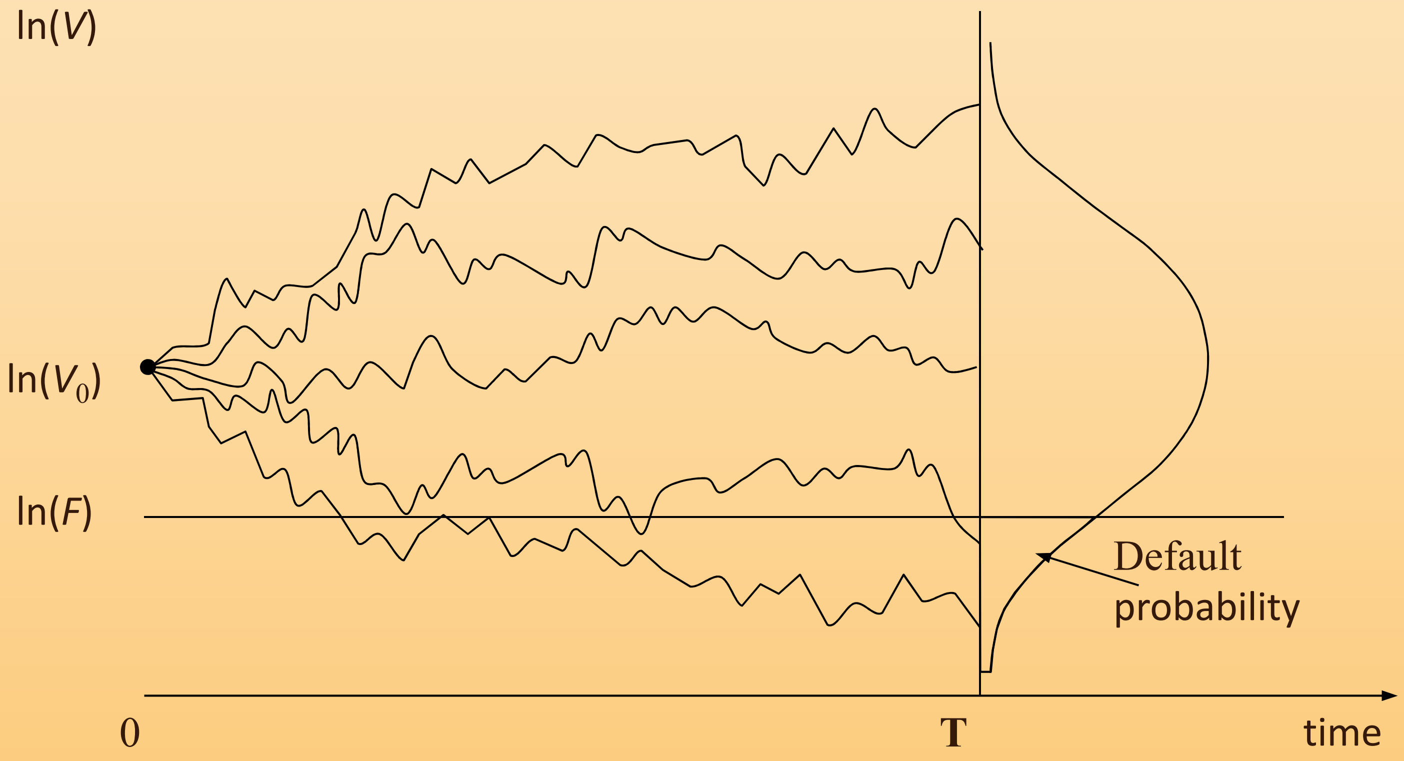
Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

- Merton(1974) applies the Black-Sholes-Merton option-pricing model to price the debt of a firm with a simple capital structure.
  - **Perfect market assumptions** in an economy that allows trading in continuous time
  - **Two types of claims**: Single homogeneous class of debt and the residual claim equity
  - The firm cannot issue new claims nor can it pay cash dividends or share purchase prior to the maturity of debt
  - The firm cannot default at any time before the bond maturity date T

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

- **Figure 11.1** illustrates different paths that the firm value process may take.



Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

- If at maturity, the firm value falls below the face value of debt, the firm cannot fully redeem the face value even by liquidating all its assets.
  - The bondholders take over the firm and shareholders get nothing.

- The payoff to stock and zero-coupon bond at maturity are shown in **Table 11.1**

	$V(T) \geq F$	**$V(T) < F$**
Stock value at T	$V(T) - F$	0
Bond value at T	F	$V(T)$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

- These payoff are mathematically given as follows:

$$S(T) = \max[0, V(T) - F] \quad (11.13)$$

$$D(T,T) = \min[V(T), F] \quad (11.14)$$

where the sum of stock and bond equals the firm value:

$$S(T) + D(T,T) = V(T) \quad (11.15)$$

- The payoff in equation makes the stock a call option on the firm's assets.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

- The bond payoff in equation 11.14 can be written alternatively as:

$$D(T,T) = \min[V(T),F] = F - \max[0, F - V(T)] \qquad (11.16)$$

The payoff represents the **default-prone zero-coupon bond** as a sum of a **long position** in a default-free zero coupon bond and a **short position** in a put option.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

- **Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework**
  - Nawalkha-Shimko et al. Models
  - Numerical Analysis
  - Relationship between Asset Duration, Bond Duration, and Stock Duration
  - Bond and Stock Durations vs. Financial and Operating Leverage
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Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

- By assuming:
  - The asset return process is given by equation 11.18
  - The bond return process is given by equation 11.15.

$$S(T) + D(T,T) = V(T) \qquad (11.15)$$

$$D_d = w_v D_v + w_p D_p \qquad (11.18)$$

where

$$w_v = \frac{[1 - N(d_1)]V(t)}{D(t,T)} \qquad (11.19)$$

$$w_p = \frac{N(d_2)P(t,T)F}{D(t,T)} \qquad (11.20)$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

- Since both weights are greater than zero and sum up to one, the duration of the default-prone zero-coupon bond **lies between** the duration of **the assets** and the duration of **the default-free zero-coupon bond**.
- Thus, if the duration of assets is greater than the duration of the default-free zero-coupon bond, then the duration of the default-prone bond is **greater** than the duration of the default-free bond.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

- Using the Merton framework, we can derive the price of a default-prone zero-coupon bond as follows:

$$D(t,T) = [1 - N(d_1)]V(t) + N(d_2)P(t,T)F \qquad (11.17)$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2}\, dz = \text{cumulative normal distribution function at } x$$

$$d_1 = \frac{-\ln\left(L(t)\right) + \frac{1}{2}V}{\sqrt{V}} \qquad d_2 = d_1 - \sqrt{V}$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

$$L(t) = \frac{P(t,T)F}{V(t)} = \text{quasi-debt ratio}$$

and

$$V = \left(\sigma_v^2 + \frac{\sigma^2}{\alpha^2} + \frac{2\rho\sigma\sigma_v}{\alpha} \right)(T-t) - B(t,T)\left(\frac{2\sigma^2}{\alpha^2} + \frac{2\rho\sigma\sigma_v}{\alpha} \right) - \frac{\sigma^2}{2\alpha^3}\left(e^{-2\alpha(T-t)} - 1 \right)$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

- **Example 11.1**

  Consider a zero-coupon bond that promises to pay $1 (F=1) in a year (T=1).

  - Volatility of firm value is $\sigma_U=0.2$
  - Risk-free rate is $r(0)=6\%$
  - Speed of mean reversion of the interest rate is $\alpha=0.2$
  - Risk-neutral long-run mean of the interest rate is $m=0.06$
  - Volatility of the interest rate process is $\sigma=0.02$
  - Correlation between the firm value and interest rate process is $\rho=-0.3$
  - Market price of risk is $\gamma=0$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

- The value of the default-free zero-coupon bond can be computed using equation 11.4 as follows:

$$B(t,T) = \frac{1-e^{-\alpha(T-t)}}{\alpha} = \frac{1-e^{-0.2(1-0)}}{0.2} = 0.9063$$

$$A(t,T) = \left(m + \frac{\sigma\gamma}{\alpha} - \frac{\sigma^2}{2\alpha^2} \right) [B(t,T) - (T-t)] - B(t,T)^2 \frac{\sigma^2}{4\alpha}$$

$$= \left(0.06 + 0 - \frac{0.02^2}{2 \times 0.2^2} \right) \times [0.9063 - 1] - 0.9063^2 \frac{0.02^2}{4 \times 0.2} = -0.0056$$

$$P(t,T) = e^{A(t,T) - B(t,T)r(t)} = e^{-0.0056 - 0.9063 \times 0.06} = 0.9418$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

- The integrated volatility V, quasi-debt ratio L(t), d_1 and d_2 in equation are given as:

$$V = \left(\sigma_v^2 + \frac{\sigma^2}{\alpha^2} + \frac{2\rho\sigma\sigma_v}{\alpha} \right)(T-t) - B(t,T)\left(\frac{2\sigma^2}{\alpha^2} + \frac{2\rho\sigma\sigma_v}{\alpha} \right) - \frac{\sigma^2}{2\alpha^3}\left(e^{-2\alpha(T-t)} - 1\right)$$

$$= \left(0.2^2 + \frac{0.02^2}{0.2^2} + \frac{2 \times (-0.3) \times 0.02 \times 0.2}{0.2} \right) \times 1$$

$$- 0.9063 \times \left(\frac{2 \times 0.02^2}{0.2^2} + \frac{2 \times (-0.3) \times 0.02 \times 0.2}{0.2} \right) - \frac{0.02^2}{2 \times 0.2^3}\left(e^{-2\times 0.2} - 1\right)$$

$$= 0.0390$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

$$L(t) = \frac{P(t,T)F}{V(t)} = \frac{0.9418 \times 1}{1.2} = 0.7848$$

$$d_1 = \frac{-\ln\left(L(t)\right) + \frac{1}{2}V}{\sqrt{V}} = \frac{-\ln(0.7848) + \frac{1}{2}0.0390}{\sqrt{0.0390}} = 1.3256$$

$$d_2 = d_1 - \sqrt{V} = 1.3256 - \sqrt{0.0390} = 1.1282$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

- The price of the default-prone bond can be computed using equation 11.17 as follows:

$$
\begin{aligned}
D(t,T) &= [1 - N(d_1)]V(t) \;+\; N(d_2)P(t,T)F \\
&= [1 - N(1.3256)] \times 0.0390 + N(1.1282) \times 0.9418 \times 1 \\
&= 0.9307
\end{aligned}
$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

- The firm value duration can be computed using equation 11.11 as follows:

$$D_v = -\frac{\sigma_v \ \rho}{\sigma} = -\frac{0.2 \times (-0.3)}{0.02} = 3$$

- Duration of the default-free bond can be computed using equation 11.6 as follows:

$$D_p = B(t,T) = 0.9063$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Nawalkha-Shimko et al. Models

- Using equations 11.18 and 11.21, the durations of the default-prone bond and the stock can be computed as follows:

$$D_d = w_v D_v + w_p D_p$$
$$= 0.1192 \times 3 + 0.9908 \times 0.9063 = 1.1560$$

$$D_s = \frac{V(t)}{S(t)} D_v - \frac{D(t,T)}{S(t)} D_d$$
$$= \frac{1.2}{1.2 - 0.9307} \times 3 - \frac{0.9307}{1.2 - 0.9307} \times 1.1560 = 9.3733$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

- **Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework**
  - Nawalkha-Shimko et al. Models
  - Numerical Analysis
    - ① Relationship between Asset Duration, Bond Duration, and Stock Duration
    - ② Bond and Stock Durations vs. Financial and Operating Leverage
    - ③ Relationship between Credit Spread Changes and Interest Rate Changes

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- In this section, we perform numerical analysis to see the role played by the firm's **asset duration** in determining the durations of the firm's default-prone bond and stock.

- We also investigate how the default-prone bond duration changes with respect to financial risk and business risk of the firm.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

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Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- The asset duration can be given as:

$$D_v = \left[\frac{D(t,T)}{V(t)}\right] D_d + \left[\frac{S(t)}{V(t)}\right] D_s \qquad (11.22)$$

- Equation 11.22 follows from balance-sheet identity and is valid even when interest and dividend payments are allowed.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- Based on empiricism, let's assume that $D_d > 0$, equation 11.22 implies the following result:

$$\text{If } D_v \le 0, \quad \text{then} \quad D_s < 0 \quad \text{and} \quad \left|D_s\right| \ge \left|\left[\frac{D(t,T)}{S(t)}\right] D_d\right| \qquad (11.23)$$

- Thus, assuming that the duration of a firm's assets is less than or equal to zero implies that the duration of its stock must be significantly negative.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- This is counterintuitive, since investors usually associate interest rate risk with bonds and not stocks.
- The **empirical evidence** implies that the duration values of most **stocks** should be either positive or close to zero, but not significantly negative.
- Hence, under realistic condition, the duration of the **assets** of most firms should not be less than or equal to zero.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

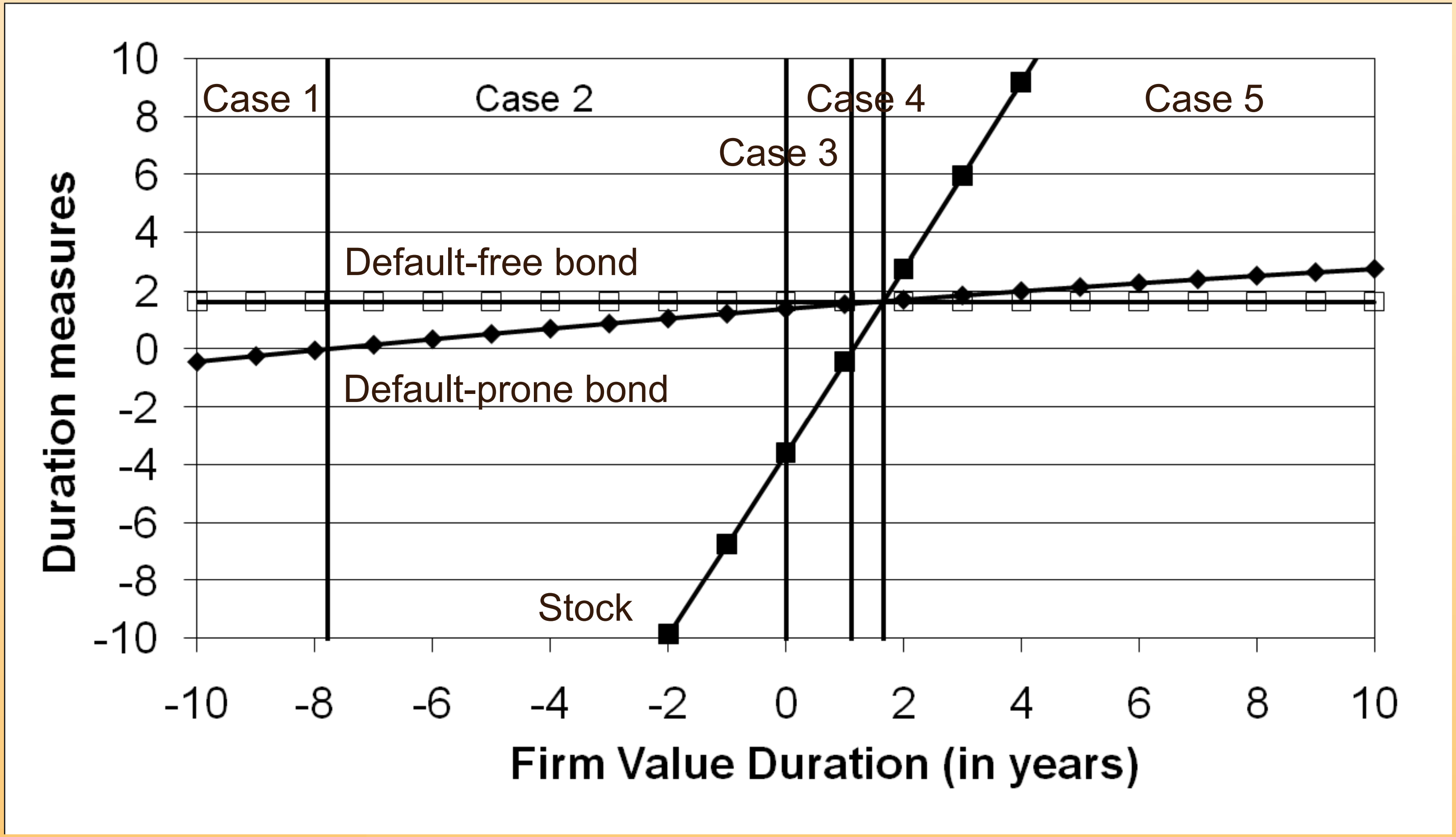
- We now assume that a firm's outstanding liabilities are given by a single non-dividend paying stock and a single default-prone zero-coupon bond.

- To analyze the relationship between **the asset duration**, the duration of **the default-prone-coupon bond**, and **the duration of the non-dividend paying stock**, we consider five cases given next in **Figure 11.2**.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

Figure 11.2



Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- A variable that plays a key role in dividing the range of asset duration values among five cases, from the most negative to the most positive, is the **quasi-debt ratio** ***L(t)***.

- To analyze these five cases, we introduce two more variables given as:

$$K_1 = \frac{N(d_2)}{N(d_1)-1}\frac{P(t,T)F}{V(t)} = \frac{N(d_2)}{N(d_1)-1}L(t) < 0 \quad (11.24)$$

$$K_2 = \frac{N(d_2)}{N(d_1)}\frac{P(t,T)F}{V(t)} = \frac{N(d_2)}{N(d_1)}L(t) > 0 \quad (11.25)$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- The five cases are given as:
  - ***Case 1:*** $D_v < K_1 D_p < 0$. Under this case:

    $D_d < 0$, $D_s < 0$, and $|D_s| > |D_d|$
  - ***Case 2:*** $K_1 D_p < D_A < 0$. Under this case:

    $D_d \geq 0$, $D_s < 0$, and $|D_s| \geq |[D(t,T)/S(t)]D_d|$
  - ***Case 3:*** $0 < D_A \leq K_2 D_P$. Under this case:

    $D_d > 0$, $D_s \leq 0$, and $|D_s| < |[D(t,T)/S(t)]D_d|$
  - ***Case 4:*** $0 < K_2 D_P < D_A < D_P$. Under this case:

    $D_d > 0$, $D_s > 0$, and $|D_s| < |D_d|$
  - ***Case 5:*** $0 < D_P < D_A$. Under this case:

    $D_d > 0$, $D_s > 0$, and $|D_s| \geq |D_d|$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- These cases consider asset durations ranging from negative values (in Case 1 and Case 2), to positive values (in Case 3, Case 4, and Case 5).
- It shows that positive duration values for assets are more consistent with empirical data on stocks and bonds.
- Note that the slope of the stock duration line is greater than the slope of the bond duration line.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- In case 1 & 2, both these cases imply significantly negative duration values for non-dividend paying stocks, which is inconsistent with the empirical studies.

- In case 3, the duration of the non-dividend paying stock must be less than or equal to zero, and the magnitude is closer to zero.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- In case 4, it's consistent with the observed negative correlation between the returns on stock portfolios and the changes in the nominal interest rates.
- In case 5, it considers high positive duration values for a firm's assets, and the magnitude of stock duration dominates the bond duration.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- **Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework**
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Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- This section assumes a **constant asset duration**, and analyzes the relationships of the default-prone bond duration and the stock duration with respect to:
  - The quasi-debt ratio *L(t)* (to measure financial leverage)
  - The standard deviation of the firm's asset return σ_V (to measure business risk)

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

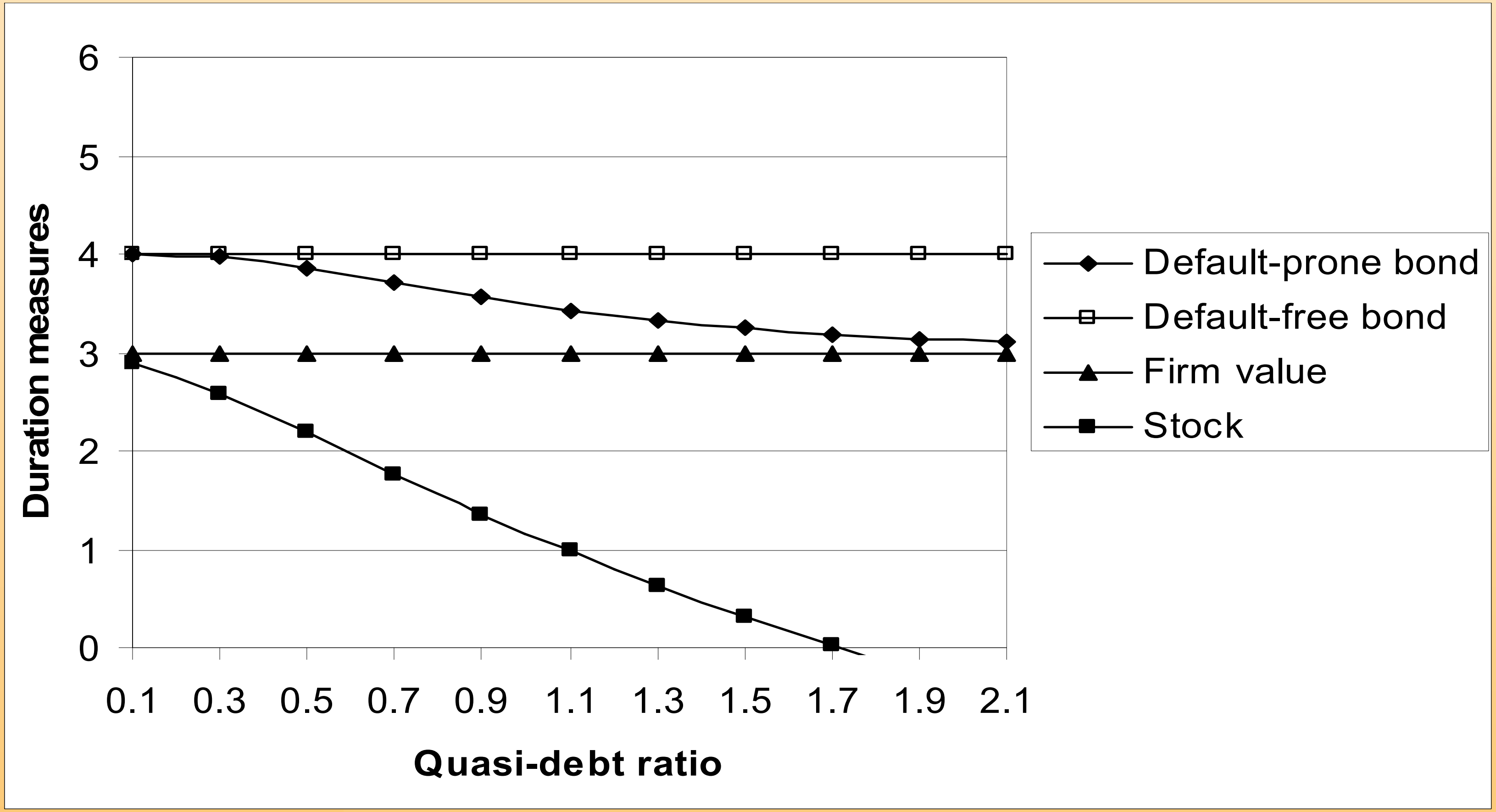
Numerical Analysis

- **Figure 11.3** illustrates the default-prone bond duration and the stock duration are decreasing functions of the financial leverage, when the duration of the firm's assets is less than the duration of the default-free bond.
- **Figure 11.4** illustrates the default-prone bond duration and the stock duration are increasing functions of the financial leverage, when the duration of the firm's assets is greater than the duration of the default-free bond.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

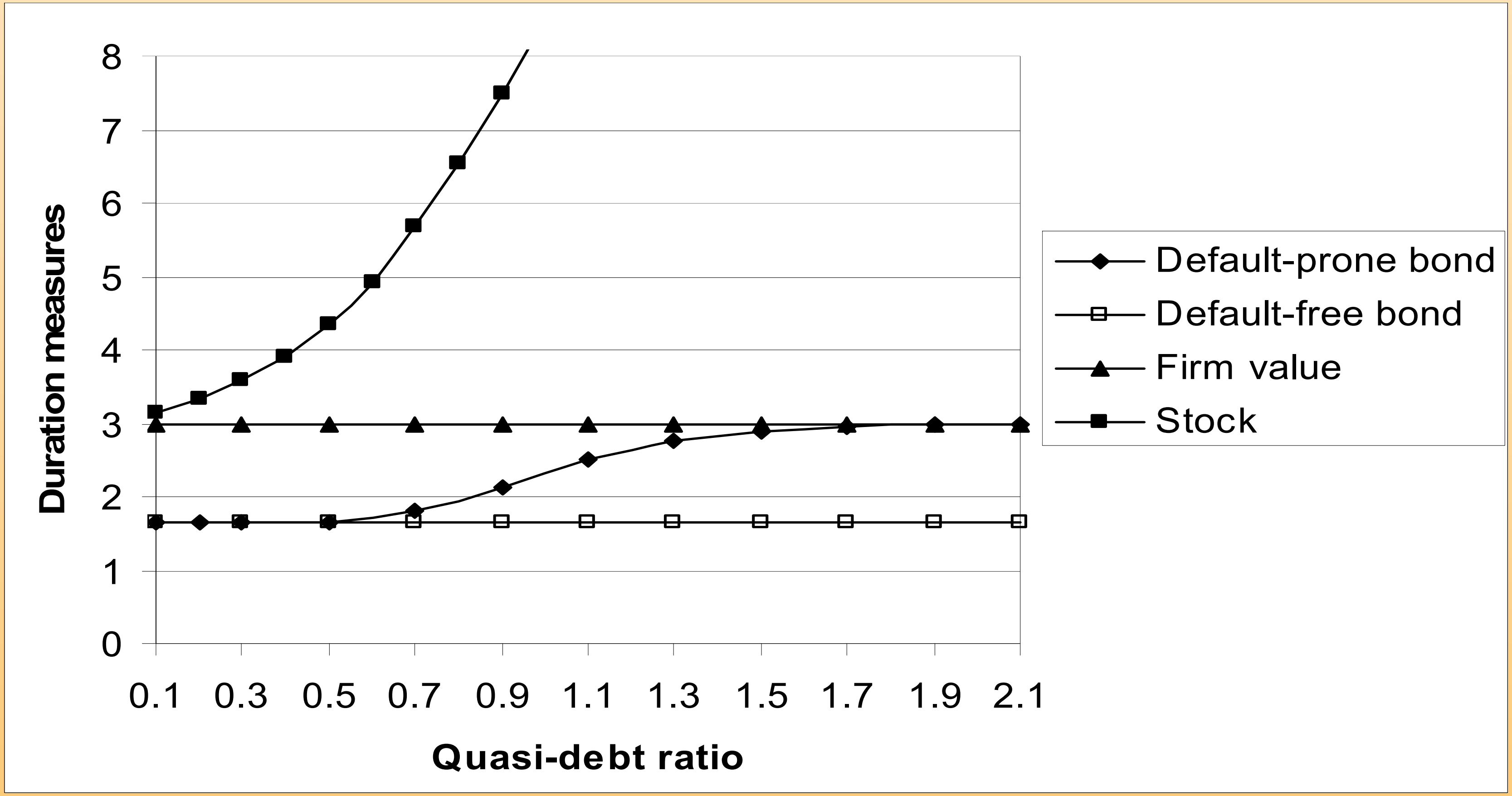
- **Figure 11.3**



Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- **Figure 11.4**



Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- These findings are consistent with the fact that at extremely **low** levels of leverage the default-prone bond must behave like the **underlying default-free bond**.

- And at extremely **high** levels of leverage the default-prone bond must behave like the **underlying assets of the firm** in the Merton [1974] framework.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

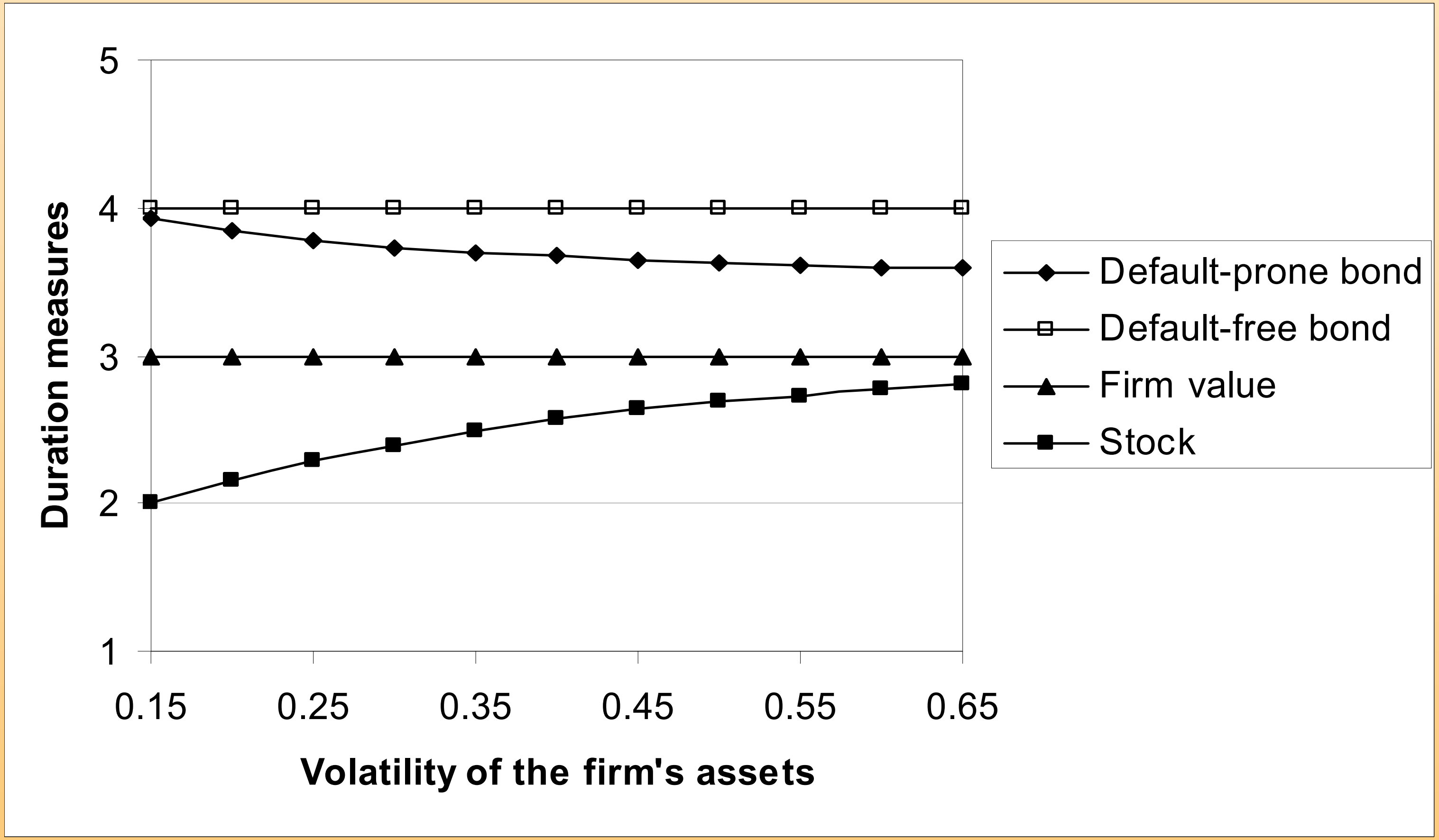
- **Figure 11.5** demonstrates that when the asset duration is lower than the duration of the default-free bond, the default-prone bond (stock) duration is a decreasing (increasing) function of the standard deviation of the firm's asset return.

- **Figure 11.6** demonstrates, when the asset duration is higher than the duration of the default-free bond, the default-prone bond (stock) duration is an increasing (decreasing) function of the standard deviation of the firm's asset return.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- **Figure 11.5**



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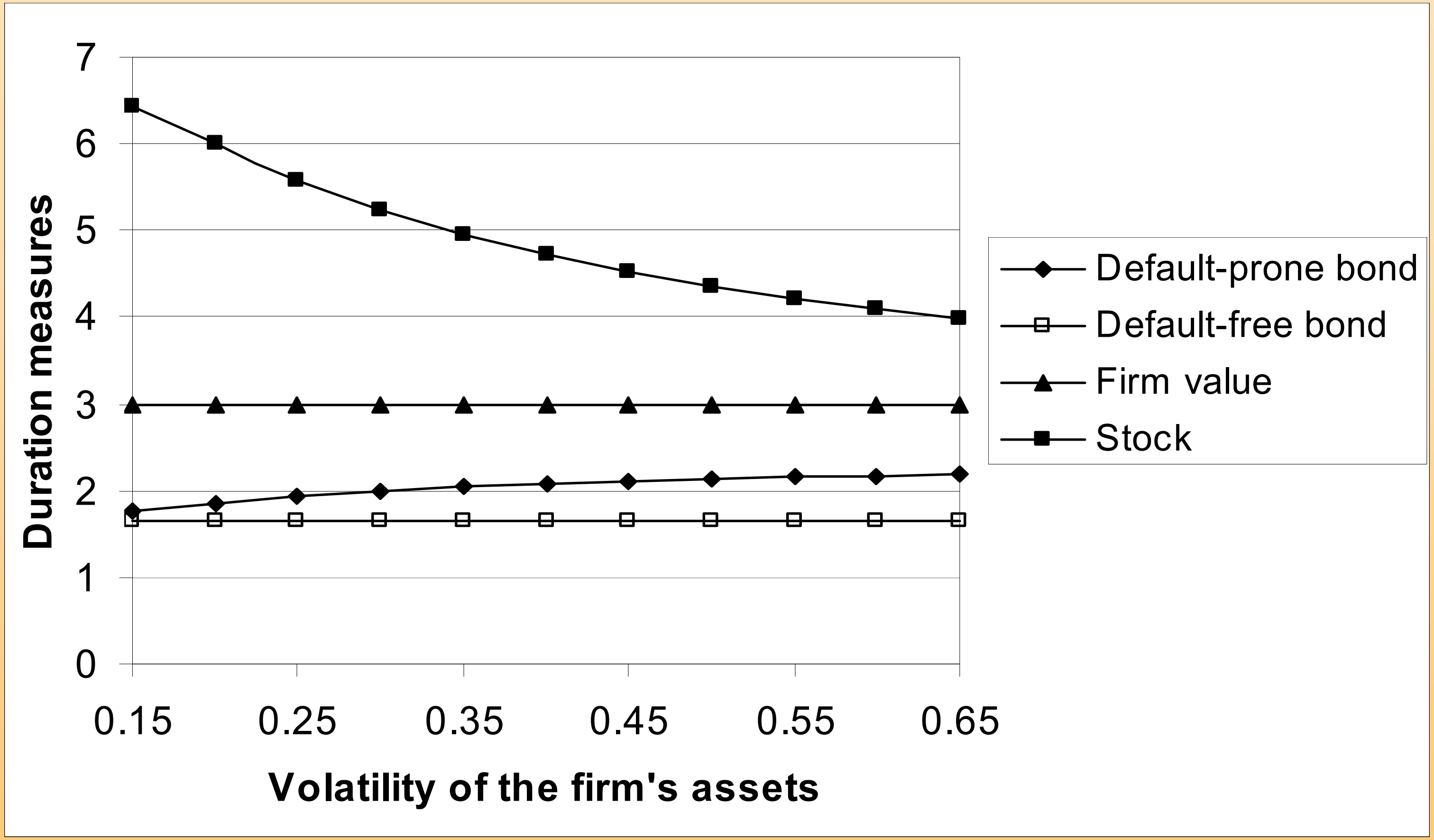
Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- **Figure 11.6**



Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- The above results provide new insights in relation to the well-known empirical result that interest rate risk is inversely related to default-risk.

- When the **assets** of firms are very **highly sensitive to interest rate risk**, such as assets of the utility industry firms or depository institutions that lend long and borrow short, **the interest rate risk of the default-prone bonds may actually increase as the default-risk increases**.
  - (consistent with Figure 11.4, Figure 11.6, and Case 5 in Figure 11.2)

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

- **Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework**
  - Nawalkha-Shimko et al. Models
  - Numerical Analysis
    ① Relationship between Asset Duration, Bond Duration, and Stock Duration
    ② Bond and Stock Durations vs. Financial and Operating Leverage
    ③ Relationship between Credit Spread Changes and Interest Rate Changes

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- A significant body of research argues that **credit spreads** and **default-free interest rates** are inversely related.
- Though the above relationship should generally hold, it can be easily demonstrated that the relationship may not hold if asset duration is significantly higher than the default-free bond's duration.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- Two important variables that determine the relationship between **changes in the short rate** and **changes in the credit spread** are:
  - – The sensitivity of the underlying assets to interest rate changes, measured by the asset duration
  - – Maturity of the default-prone bond, which determines the duration of the equivalent default-free bond

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- As a general result, **credit spreads:**
  - of **shorter maturity** bonds issued by corporations with highly interest rate sensitive assets will *increase*
  - of **longer maturity** bonds issued by corporations that have assets with low or negative interest rate sensitivity will *decrease*

,…in response to an increase in the default-free short rate.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- The credit spread is defined as the difference between the yields-to-maturity of the default-prone zero-coupon bond and the default-free zero coupon bond:

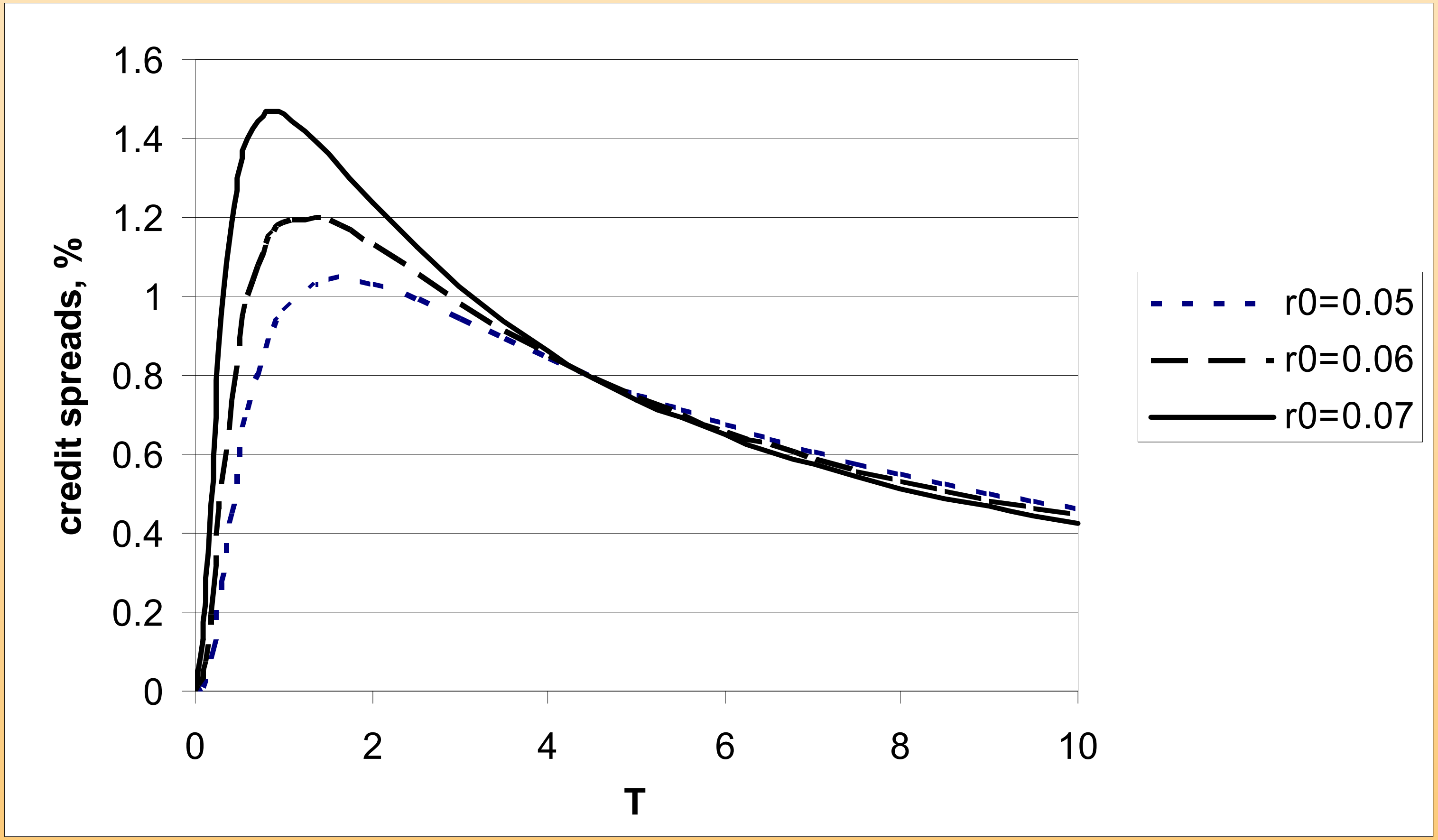
$$s(t,T) = y_c(t,T) - y(t,T) = -\frac{1}{T-t}\ln\left(N(d_2) + \frac{(1-N(d_1))}{L(t)} \right) \quad (11.26)$$

- **Figure 11.7** demonstrates the relationship between the default-free short rate and the term structure of credit spread.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- **Figure 11.7**



Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- We choose the following base parameter values:

$$V(0)=\$120,\ F = \$100,\ r(0) = 6\%,$$

$$\sigma_V = 20\%,\ \alpha = 0.2,\ m = 6\%,\ \sigma = 2\%,$$

$$\rho = -30\%,\ \gamma = 0$$

- Given these parameter values, the asset duration equals:

$$D_V = -\frac{\sigma_V \rho}{\sigma} = -\frac{0.2 \times (-0.3)}{0.02} = 3 \qquad (11.27)$$

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- Given the asset duration equals 3, when the short rate increases by 1%, the firm value is expected to decrease by 3%.
- The credit spreads generated using the short rate of **7%** assume that the firm value decreases to its conditional expected value of \$120 – \$120 × 3%= \$116.4
- The credit spreads generated using the short rate of **5%** assume that the firm value increases to its conditional expected value of \$120 + \$120 × 3%= \$123.6.

Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework

Numerical Analysis

- It can be seen from that when the short rate rises to 7%, credit spreads increase for shorter maturities, but decrease for longer maturities.
- Similarly, when the short rate falls to 5%, credit spreads decrease for shorter maturities, but increase for longer maturities.
- The cross-over point is around 4.581 years. The point at which the default-free bond duration equals the asset duration of 3 years.

Chapter 11:
Duration Models for Default-Prone Securities

- Introduction
- Pricing and Duration of a Default-Free Zero-Coupon Bond Under The Vasicek Model
- The Asset Duration
- Pricing and Duration of a Default-Prone Zero-Coupon Bond: The Merton Framework
- Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

- The Merton [1974] framework is difficult to apply to coupon-paying bonds, since a default-prone coupon bond cannot be treated as a **portfolio** of default-prone zero-coupon bonds.
  - Because default on any given coupon payment date implies a default on future coupon payments.

- A default-prone coupon bond is not a **linear sum** of the default-prone zero-coupon bonds, thus **compound option models** must be used, since future coupons are paid **conditional**.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

- The compound option models such as Geske has two **limitations**:
  - First, it does not allow the firm to buy back or issue more debt
  - Second, it assumes that assets are divided among the liability holders according to strict absolute priority rules when the firm defaults

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

- The **first passage models** that overcome the limitations of both the Merton model and the Geske's extension of the Merton model.

- The first stream of research was composed by Black and Cox (1976).
  - The probability of default is given as the **first passage probability** of the firm value hitting the given default threshold.
  - They assume **constant interest rates** and **absolute priority rules**

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

- Longstaff and Schwartz (1995), who allow stochastic interest rates as well as violations of the absolute priority rules.

- In this section we are going to give an introduction to the models of **Black and Cox**, and **Longstaff & Schwartz**.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

- **Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models**
  - Black and Cox Model
  - Longstaff and Schwartz Model
  - Duration of a Default-Prone Bond
  - Numerical Analysis

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Black and Cox Model

- Merton assumes that the firm can default only at maturity if the asset value falls below the face value of debt.
  - This is unrealistic and creates highly perverse, gambling incentives for stockholders, when asset value plummets.

- Black and Cox specify a **lower default threshold boundary**, which is the level at which the firm is forced into bankruptcy or reorganization.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Black and Cox Model

- The lower default threshold level can be given as a **safety covenant**, which the bondholders are entitled with the right to force the firm into bankruptcy if its value falls to a pre-specified level.

- Black and Cox assume that the stockholders receive a continuous dividend payment proportional to the firm value, aV.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Black and Cox Model

- The firm value process is given as follows:

$$\frac{dV(t)}{V(t)} = (\mu - a)dt + \sigma_v dZ_v(t) \qquad (11.28)$$

where μ is the expected rate of return on the firm's assets

- Black and Cox suggest using a time-dependant default-threshold of an exponential form given as follows:

$$K(t) = K\,e^{-\gamma(T-t)} \qquad (11.29)$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Black and Cox Model

- The terminal payoffs in the Black and Cox model are given in **Table 11.2**

	$V(T) \geq F$	$K \leq V(T) < F$
Stock value at *T*	$V(T) - F$	0
Bond value at *T*	F	$V(T)$

- The firm value V(t) is always greater or equal to the default threshold K(t), which is an **absorbing barrier**.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Black and Cox Model

- The absorbing barrier can be translated into the following boundary conditions for the stock and the bond:

$$S(T) = \max\left[V\left(T\right) - F, 0\right] \quad (11.30)$$

$$D(T,T) = \min\left[V\left(T\right), F\right] \quad (11.31)$$

- Since the firm may default at any time, it is necessary to specify two more conditions:

$$\text{If } V(t) = K(t), \text{ then } S(t) = 0,$$

$$\text{and } D(t,T) = K(t) = Ke^{-\gamma(T-t)} \quad (11.32)$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Black and Cox Model

- Black and Cox obtain the following solution to the zero-coupon bond price:

$$
\begin{aligned}
D(t,T) = Fe^{-r(T-t)}\left[N(z_1) - y^{2\theta-2}N(z_2)\right] \\
+Ve^{-a(T-t)}[N(z_3) + y^{2\theta}N(z_4) \\
+ y^{\theta+\varsigma}e^{a(T-t)}N(z_5) + y^{\theta-\varsigma}e^{a(T-t)}N(z_6) \\
- y^{\boxed{\theta+\eta}}N(z_7) - y^{\theta-\eta}N(z_8)]
\end{aligned}
\qquad (11.33)
$$

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Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

- **Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models**
  - Black and Cox Model
  - Longstaff and Schwartz Model
  - Duration of a Default-Prone Bond
  - Numerical Analysis

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

- The Longstaff and Schwartz (1995) model extends the Black and Cox model in two important ways:
  - **First**, this model allows interaction between default risk and interest rate risk, by allowing for stochastic interest rates.
  - **Second**, this model allows for deviations from strict absolute priority rules.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

- Longstaff and Schwartz consider a firm that issues a default-prone bond with a periodic coupon payment *C*, and a face value *F* maturing at time *T*.
- They assume that a default threshold value *K* exists at which financial distress occurs.
- As long as the firm value *V(t)* is greater than *K*, the firm continues to meet its interest payments to all bondholders.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

- If $V(t)$ reaches K, the firm defaults on all its bond obligations, simultaneously.

- When financial distress is triggered ($V(t) = K$), the total assets are allocated to the various classes of bond claimants through some form of **corporate restructuring,** such as…
  - Chapter 11 reorganization or liquidation
  - Chapter 7 liquidation
  - Private debt restructuring

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

- Longstaff and Schwartz assume that upon reaching the default threshold *K*, for every $1 of face value of a given bond, the bondholder receives **1- *w***, at maturity.

- **w** is the value of **average write-down**. The constraint is that the total settlement of all claimants cannot exceed the lower threshold value *K*.

- Longstaff and Schwartz also assume that the firm value is **independent of its capital structure**. Due to this, the actual bondholders' recovery in case of default is <u>not</u> related to the face value of the bond.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

- A much simpler solution to Longstaff and Schwartz model based on approximation of **one-dimensional integral** was provided by Mueller [2002] and is given as follows:

$$D(t,T) = P(t,T)F\left(1 - wQ(t,T)\right) \qquad (11.35)$$

where P(t,T) is the price of the default-free bond, F is the face value of the bond, and Q(t,T) is the first passage probability of default.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

- **Example 11.2**

  Consider a zero coupon bond that promises to pay \$100 (i.e., F = 100) in one year (i.e., T = 1). The current value of the firm is \$120 (i.e., $V(0)$ = 120).

$$K = \$100,\ w = 0.4,\ \sigma_v = 0.2,\ r_0 = 6\%,\ \alpha = 0.2,$$
$$m = 0.06,\ \sigma = 0.02,\ \rho = -0.3$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

The computation of the first passage probability in equation 11.36 requires dividing the maturity of T years into $\boldsymbol{n}$ small intervals of Δ each.

For expositional purpose, we assume that $n = 5$, $\Delta = T/n$.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

$$Q(t,T)=\sum_{i=1}^{n} q_i; \quad q_1=\frac{N(a_1)}{N(b_{1,1})}$$

$$q_i=\frac{N(a_i)-\sum_{j=1}^{i-1} N(b_{i,j})\times q_j}{N(b_{i,i})}, \qquad i=2,\ldots,n \quad (11.36)$$

$$a_i=\frac{\ln K-M(i\Delta,T)}{\sqrt{S(i\Delta)}}; \quad b_{i,j}=\frac{\ln K-M(i\Delta,t_j,\ln K,T)}{\sqrt{S(i\Delta,t_j)}}$$

$$t_j\in[(j-1)\Delta, j\Delta]$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

The first row of Table 11.3a provides the value of a_1 and $b_{1,1}$ computed using the formulas given above.

The second row of this table gives the standard Normal distribution function evaluated at a_1 and $b_{1,1}$

	a_1	$b_{1.1}$
	-2.1362	-0.0733
N(.)	0.0163	0.4708

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

Given the values of $N(a_1)$ and $N(b_{1,1})$, the value of q_1 can be computed using equation, as follows:

$$q_1 = \frac{N(a_1)}{N(b_{1,1})} = \frac{0.0163}{0.4708} = 0.0347$$

Using similar computations in tables 11.3b, 11.3c, 11.3d, and 11.3e, the values of q_2, q_3, q_4, and q_5 can be computed as follows.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

	a_2	$b_{2,1}$	$b_{2,2}$
	-1.5793	-0.1267	-0.0729
N(.)	0.0571	0.4496	0.4710

$$q_2 = \frac{N(a_2) - N(b_{2,1}) \times q_1}{N(b_{2,2})} = \frac{0.0571 - 0.4496 \times 0.0347}{0.4710} = 0.0882$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

	a_3	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$
	-1.3454	-0.1631	-0.1259	-0.0724
N(.)	0.0892	0.4352	0.4499	0.4711

$$q_3 = \frac{N(a_3) - \left(N(b_{3,1}) \times q_1 + N(b_{3,2}) \times q_2\right)}{N(b_{3,3})} =$$

$$= \frac{0.0892 - \left(0.4352 \times 0.0347 + 0.4499 \times 0.0882\right)}{0.4711}$$

$$= 0.0732$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

	a_4	$b_{4,1}$	$b_{4,2}$	$b_{4,3}$	$b_{4,4}$
	-1.2134	-0.1925	-0.1621	-0.1251	-0.0719
N(.)	0.1125	0.4237	0.4356	0.4502	0.4713

$$q_4 = \frac{N(a_4) - \left(N(b_{4,1}) \times q_1 + N(b_{4,2}) \times q_2 + N(b_{4,3}) \times q_3\right)}{N(b_{4,4})}$$

$$= \frac{0.1125 - \left(0.4237 \times 0.0347 + 0.4356 \times 0.0882 + 0.4502 \times 0.0732\right)}{0.4713}$$

$$= 0.0561$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

	a_5	$b_{5,1}$	$b_{5,2}$	$b_{5,3}$	$b_{5,4}$	$b_{5,5}$
	-1.1282	-0.2177	-0.1913	-0.1610	-0.1243	-0.0715
N(.)	0.1296	0.4138	0.4242	0.4360	0.4506	0.4715

$$q_5 = \frac{N(a_5) - \left(N(b_{5,1}) \times q_1 + N(b_{5,2}) \times q_2 + N(b_{5,3}) \times q_3 + N(b_{5,4}) \times q_4\right)}{N(b_{5,5})}$$

$$= \frac{0.1296 - \left(0.4138 \times 0.0347 + 0.4242 \times 0.0882 + 0.4360 \times 0.0732 + 0.4506 \times 0.0561\right)}{0.4715}$$

$$= 0.0439$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

Given the values of q_1, …q_5, we can compute the first passage probability of default, $Q(0,1)$ as follows:

$$Q(0,1) = \sum_{i=1}^{5} q_i$$
$$= 0.0347 + 0.0882 + 0.0732 + 0.0561 + 0.0439$$
$$= 0.2960$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

Using the closed-form of $P(t, T)$ in equation11.4, we get:

$$P(0,1) = e^{-0.0056 - 0.9063 \times 0.06} = 0.941$$

Substituting the above values of $Q(0,1)$ and $P(0,1)$ and the parameters, the price of the default-prone bond, $D(0,1)$, is given as follows:

$$D(0,1) = 0.9418 \times 100 \times \left(1 - 0.4 \times 0.2960\right) = \$83.03$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

In the Longstaff and Schwartz model, the value of a coupon-bond can be given as **a portfolio of zero-coupon bonds**.

Hence the value of the coupon bond can be written as:

$$D_{coup}(t,T) = P(t,T)F\left(1 - wQ(t,T)\right) + \sum_{k=1}^{N} P(t,t_k)C\left(1 - w_{coup}Q(t,t_k)\right) \qquad (11.42)$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

In practice only the portion of principal payment is redeemed and coupon payments are written down completely (w_{coup} = 1).

The yield to maturity of a corporate coupon bond, $y_c(t, T)$ can be obtained implicitly by solving the following equation:

$$D_{coup}(t,T) = Fe^{-y_c(t,T)\times T} + \sum_{k=1}^{N} Ce^{-y_c(t,T)\times t_k} \quad (11.43)$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Longstaff and Schwartz Model

The yield to maturity of a default-free coupon bond, $y(t, T)$ can be obtained implicitly by solving the following equation:

$$P_{coup}(t,T) = Fe^{-y(t,T)\times T} + \sum_{k=1}^{N} Ce^{-y(t,T)\times t_k} \qquad (11.44)$$

The credit spread is defined as the difference between the yields to maturity of the corporate coupon bond and the riskless coupon bond:

$$s(t,T) = y_c(t,T) - y(t,T) \qquad (11.45)$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

- **Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models**
  - Black and Cox Model
  - Longstaff and Schwartz Model
  - Duration of a Default-Prone Bond
  - Numerical Analysis

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Duration of a Default-Prone Bond

- In this section are going to show another reason for the **misunderstanding of the duration**, which is obtained using a simplistic comparative static analysis.

- Bt taking the partial derivative of equation 11.35, we get:

$$\frac{\partial D(t,T)}{\partial r(t)} = \frac{\partial P(t,T)}{\partial r(t)}(1 - wQ(t,T)) - wP(t,T)\frac{\partial Q(t,T)}{\partial r(t)} \qquad (11.46)$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Duration of a Default-Prone Bond

- Dividing both side of the above equation by *–D(t,T),* duration would be defined as follows:

$$
\begin{aligned}
D_d &= -\frac{\partial D(t,T)/\partial r(t)}{D(t,T)} \\
&= -\frac{\partial P(t,T)/\partial r(t)(1-wQ(t,T)}{P(t,T)(1-wQ(t,T))} - \frac{wP(t,T)\partial Q(t,T)/\partial r(t)}{P(t,T)(1-wQ(t,T))} \qquad (11.47) \\
&= -\frac{\partial P(t,T)/\partial r(t)}{P(t,T)} + \frac{w}{1-wQ(t,T)}\partial Q(t,T)/\partial r(t) D_p \\
&\quad + \frac{w}{1-wQ(t,T)}\partial Q(t,T)/\partial r(t)
\end{aligned}
$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Duration of a Default-Prone Bond

- However, using a comparative static analysis would require evaluating the partial derivative of $Q(t,T)$ with respect to $r(t)$, while keeping $x(t) = \ln V(t)$ constant.

- Doing this implicitly assumes a **zero value for the asset duration** since the firm value is kept constant even though the instantaneous short rate changes.
  - It will lead to an underestimation of the true sensitivity.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Duration of a Default-Prone Bond

- The above inappropriate use is the reason that some researches say the duration of the default-prone bond is **always lower** than the duration of the equivalent default-free bond.
- The following we suggest that an alternative numerical approach to computing the duration of the default-prone bond under the Longstaff and Schwartz model…

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Duration of a Default-Prone Bond

- The value of the default-prone zero-coupon bond D(t,T) in equation 11.35 can be written as:

$$D(t,T)=D(V(t), r(t), t, T) \qquad (11.48)$$

- Let the short rate after this instantaneous change be given as $r'(t) = r(t) + \Delta r(t)$. The *expected* change in the firm's asset value $V(t)$, caused by the change in the short rate is given as follows:

$$\Delta V(t) = -V(t) D_v \Delta r(t) \qquad (11.49)$$

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Duration of a Default-Prone Bond

- Define the new firm value as $V'(t) = V(t) + \Delta V(t)$.
- The duration of the default-prone zero-coupon bond can be now defined as follows:

$$D_d = \frac{\Delta D(V(t), r(t), r, T) / D(V(t), r(t), t, T)}{\Delta r(t)} \qquad (11.51)$$

- Since this approach accounts for the asset duration defined in equation 11.11 , it gives an accurate estimate of the default-prone bond duration.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

- **Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models**
  - Black and Cox Model
  - Longstaff and Schwartz Model
  - Duration of a Default-Prone Bond
  - Numerical Analysis

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Numerical Analysis

- This section investigates how interest rate risk of a default-prone bond is related to the firm's **business risk and financial risk** with **exogenous** specification of **K** and **w**.
- Assume constant duration, and consider the relationship of the default-prone bond duration with respect to the σ_V and quasi-debt ratio **L(t)**.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

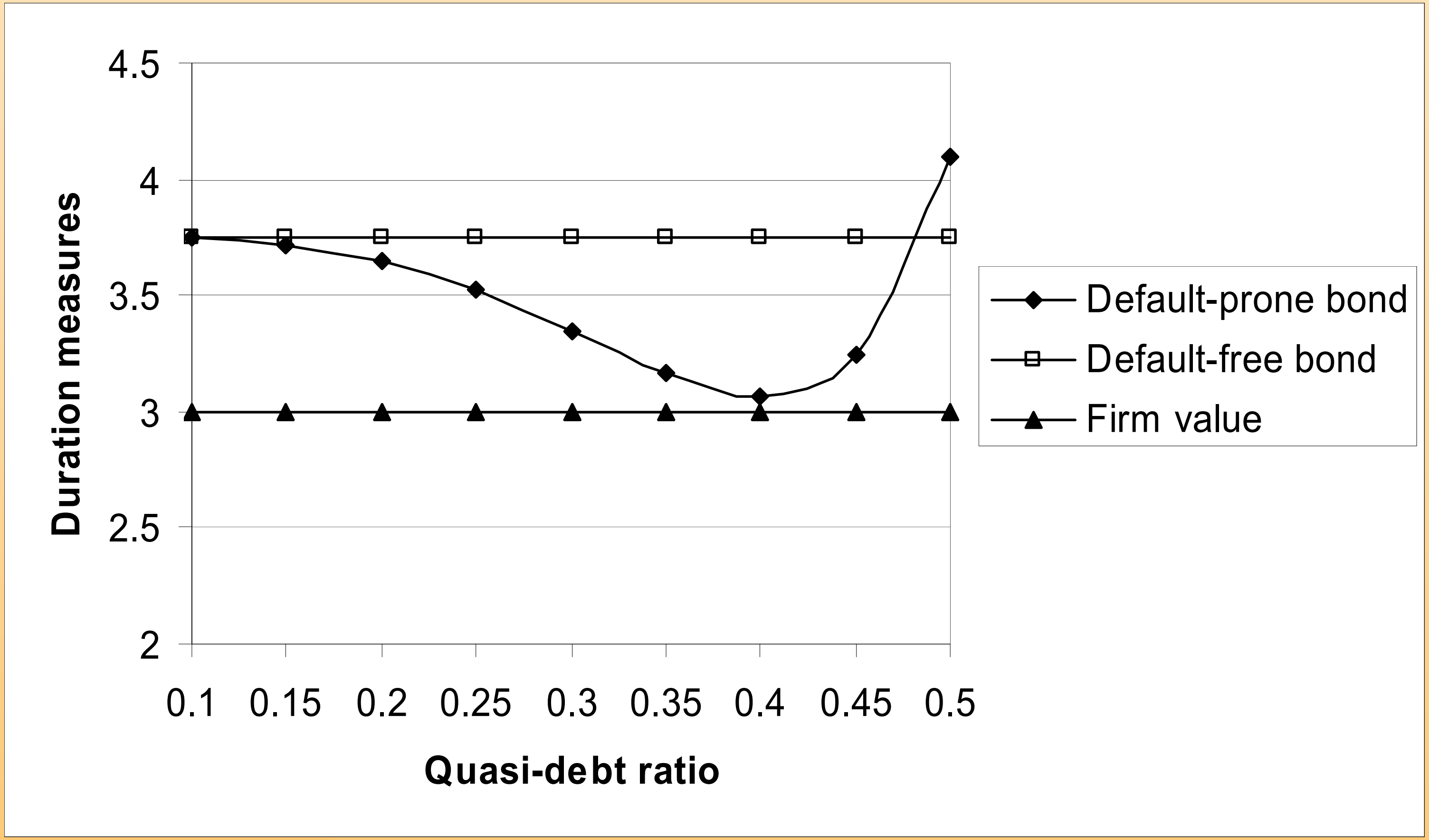
Numerical Analysis

- Consider a 6% annual coupon bond, which matures in two years or in 12 years.
- And r(0)=6%, α = 0.2, m = 6%, σ = 2%, $V(0)$ = \$200, F = \$100, K = \$80, σ_V = 20%, ρ = -0.3, w = 0.4, w_c = 1.
- **Figure 11.8** illustrates that the default-prone bond duration is initially a **decreasing** function of the **financial leverage** when the duration of the firm's assets is less than the duration of the default-free bond.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Numerical Analysis

- **Figure 11.8**



Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Numerical Analysis

- However, the duration of the default-prone bond **increases** after the initial decline.
- An almost certain default makes the default-prone coupon bond behave more like a **default-free zero-coupon bond**.
- This explains why at a **high quasi-debt ratio**, the duration of the default-prone coupon bond exceeds even the duration of the default-free coupon bond.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

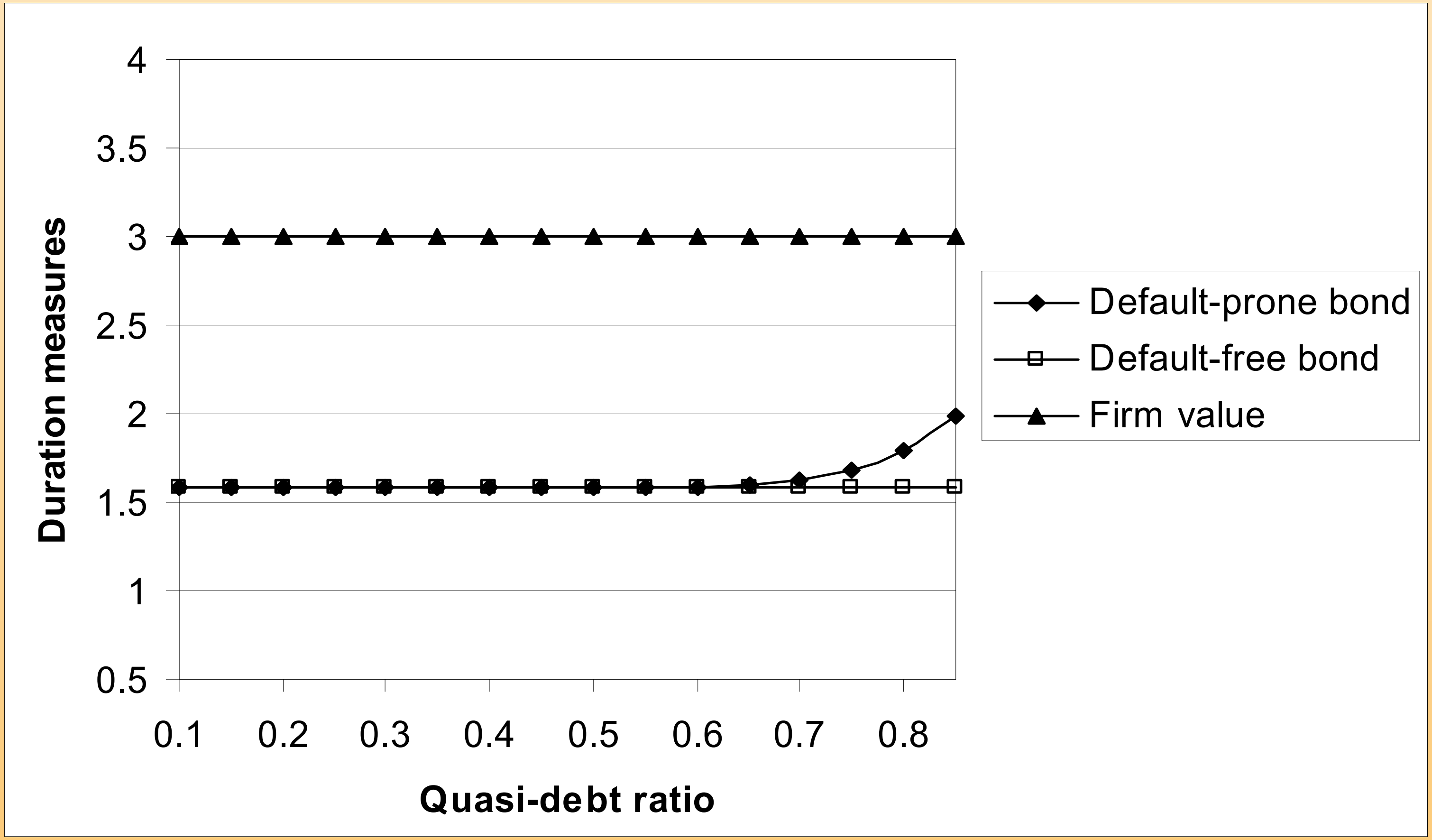
Numerical Analysis

- As shown in **Figure 11.9**, only at extremely high financial leverage the duration of the default-prone bond starts increasing towards the asset duration value.
- The duration of the default-prone coupon bond is not as sensitive to financial leverage when the asset duration is higher than the duration of the default-free coupon bond.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Numerical Analysis

- **Figure 11.9**



Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

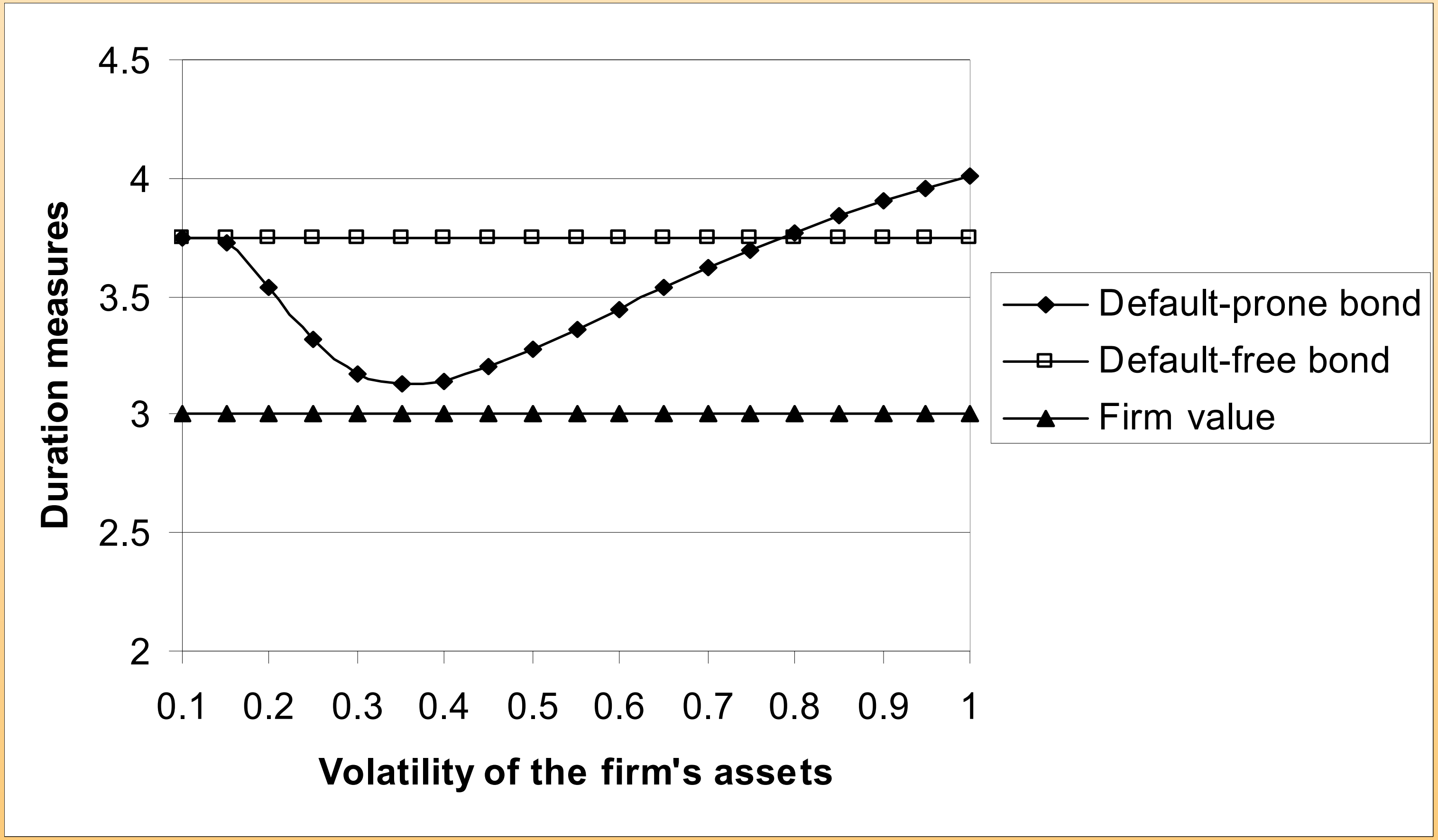
Numerical Analysis

- **Figure 11.10** illustrates the relationship between the duration of the default-prone coupon bond and **firm's volatility**, when the asset duration is lower than the default-free bond duration.
- Higher volatility makes default more certain, and an almost certain default makes the default-prone coupon bond behave more like a default-free zero-coupon bond.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Numerical Analysis

- **Figure 11.10.**



Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

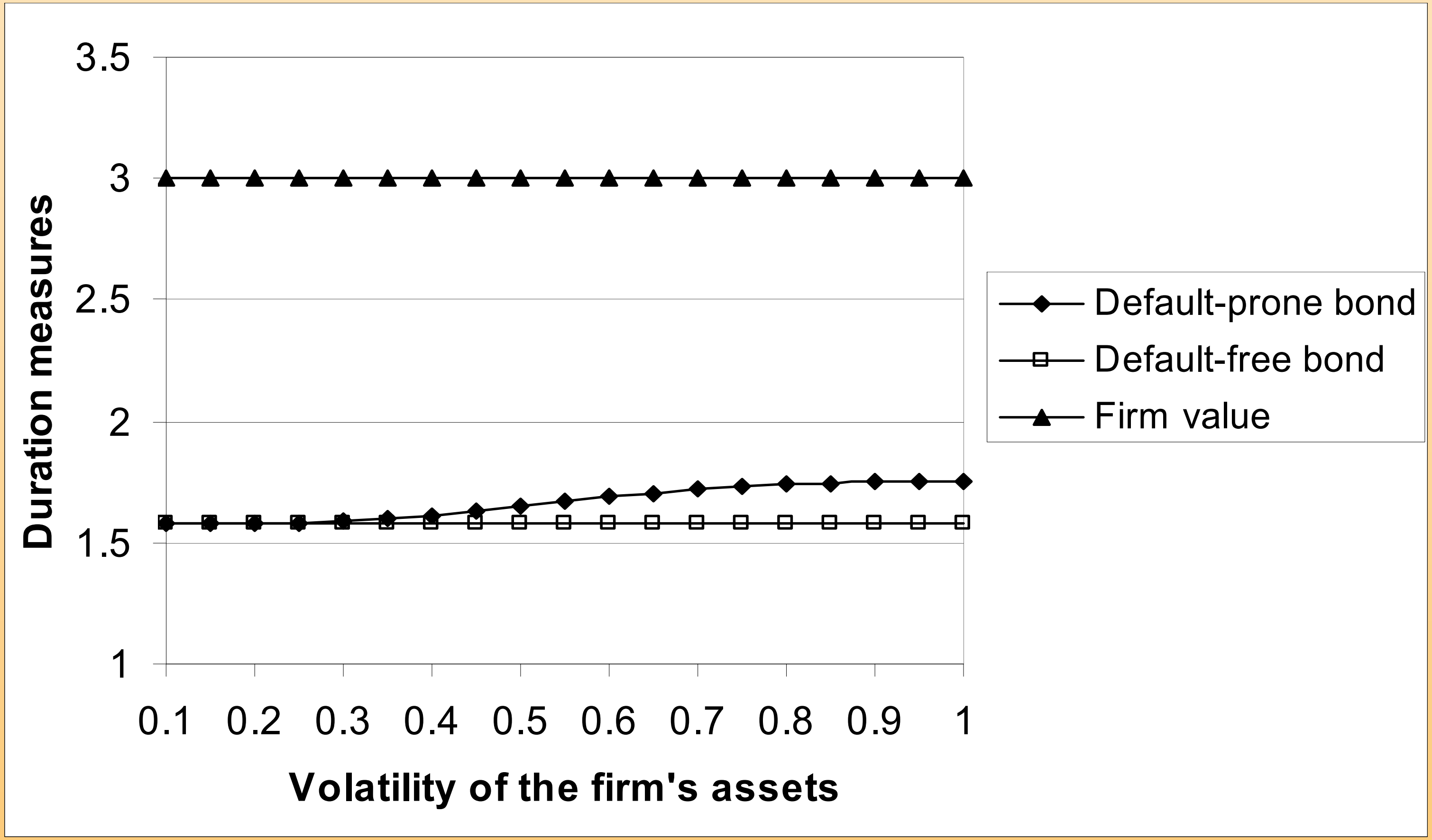
Numerical Analysis

- **Figure 11.11** illustrates the relationship between the duration of the default-prone coupon bond and firm's volatility, when the asset duration is higher than the default-free bond duration.
- The duration of the default-prone coupon bond is not as sensitive to business risk when the asset duration is higher than the duration of the default-free coupon bond.

Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Numerical Analysis

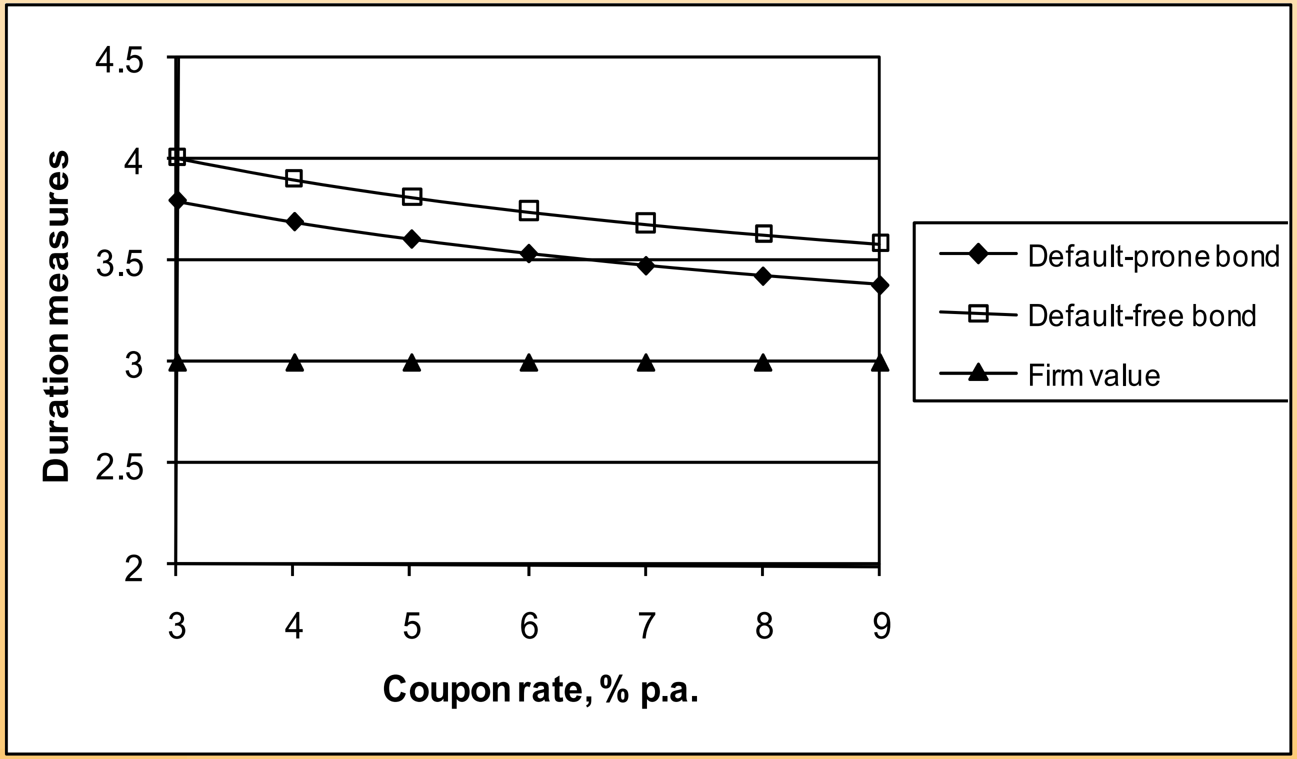
- **Figure 11.11**



Pricing and Duration of a Default-Prone Coupon Bond: The First Passage Models

Numerical Analysis

- **Figure 11.12** demonstrates that the duration of the default-prone coupon bond decreases with an increase in the annual coupon rate.



Interest Rate Risk Modeling
The Fixed Income Valuation Course

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